

Quest for Fast Partial Search Algorithm

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A quantum algorithm can find a target item in a database faster than a classical algorithm. One can trade accuracy for speed and find a part of the database (a block) containing the target item even faster, this is partial search. We consider different partial search algorithms and suggest the optimal one. Efficiency of an algorithm is measured by number of queries to the oracle.

I. INTRODUCTION

Database search has many applications and is used widely. Grover discovered a quantum algorithm that searches faster than a classical algorithm [1]. It consists of repetition of the Grover iteration \hat{G}_1 , which operates on the computational quantum states. The number of repetitions is:

$$j_{\text{full}} = \frac{\pi}{4} \sqrt{N} \quad (1)$$

for a database with large number of entries N . After j_{full} the algorithm finds the target item. For more details, see also [2–4]. Below we shall call \hat{G}_1 a global iteration.

Sometimes it is sufficient to find an approximate location of the target item. A partial search considers the following problem: a database is separated into K blocks, of a size $b = N/K$. We want to find a block with the target item, not the target item itself. Such partial search was first introduced by Mark Heiligman in [5], as a part of algorithm for list matching. We can think of partial search in following terms: an exact address of the target item is given by a sequence of n bites, but we want to find only first k bites ($k < n$). Fast quantum algorithm for a partial search was found by Grover and Radhakrishnan in [6]. They showed that classical partial search takes $\sim (N - b)$ queries, but quantum algorithm takes only $\sim (\sqrt{N} - \text{coeff}\sqrt{b})$ queries. It uses several global iterations $\hat{G}_1^{j_1}$ and then several local iteration $\hat{G}_2^{j_2}$, see (8). Local searches are Grover iterations [searches] in each individual block made in each block separately in parallel. Grover-Radhakrishnan algorithm was improved and simplified in [7]. The number of queries to the oracle in this algorithm was minimized by in [8], the coeff was maximized. Below we shall explain minimized version of Grover-Radhakrishnan algorithm. We shall call it GRK algorithm. In this paper we consider three other versions of partial search algorithm. They use different sequences of global and local searches: local-global, global-local-global and local-global-local. We prove that GRK version still uses minimal number of queries to the oracle. We conjecture that *GRK algorithm is optimal among all partial search algorithms, which consist of arbitrary sequence of local and global searches.*

The plan of the paper is as follows. In the next section we remind the Grover algorithm. After this we formulate minimized version of Grover-Radhakrishnan algorithm [GRK algorithm]. In the rest of the paper we consider other partial search algorithms. We arrive at the conclusion that GRK uses minimal number of queries comparing to other algorithms.

II. PARTIAL SEARCH

A. Global Iterations

First let us remind the full Grover search. We shall consider a database with one target item. The aim of the Grover algorithm is to identify a target state $|t\rangle$ among an unordered set of N states. This is achieved by repeating global iteration which is defined in terms of two operators. The first changes the sign of the target state $|t\rangle$ only:

$$\hat{I}_t = \hat{I} - 2|t\rangle\langle t|, \quad \langle t|t\rangle = 1, \quad (2)$$

where \hat{I} is the identity operator. The second operator,

$$\hat{I}_{s_1} = \hat{I} - 2|s_1\rangle\langle s_1|, \quad (3)$$

changes the sign of the uniform superposition of all basis states $|s_1\rangle$,

$$|s_1\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle, \quad \langle s_1 | s_1 \rangle = 1. \quad (4)$$

The **global iteration** is defined as a unitary operator

$$\hat{G}_1 = -\hat{I}_{s_1} \hat{I}_t. \quad (5)$$

We shall use eigenvectors of \hat{G}_1 :

$$\hat{G}_1 |\psi_1^\pm\rangle = \lambda_1^\pm |\psi_1^\pm\rangle, \quad \lambda_1^\pm = \exp[\pm 2i\theta_1], \quad |\psi_1^\pm\rangle = \frac{1}{\sqrt{2}} |t\rangle \pm \frac{i}{\sqrt{2}} \left(\sum_{\substack{x=0 \\ x \neq t}}^{N-1} \frac{|x\rangle}{\sqrt{(N-1)}} \right). \quad (6)$$

They were found in [4], where the angle θ_1 is defined by

$$\sin^2 \theta_1 = \frac{1}{N}. \quad (7)$$

B. GRK Algorithm for Partial Search

The first version of partial search was found in [6]. The algorithm uses j_1 global iteration and j_2 local iterations. **LOCAL ITERATIONS** are Grover iterations for each block:

$$\hat{G}_2 = -\hat{I}_{s_2} \hat{I}_t. \quad (8)$$

\hat{I}_t is given by (2), but \hat{I}_{s_2} is different. In one block it acts as:

$$\hat{I}_{s_2}|_{block} = \hat{I}|_{block} - 2|s_2\rangle\langle s_2|, \quad |s_2\rangle = \frac{1}{\sqrt{b}} \sum_{\text{one block}} |x\rangle \quad (9)$$

In the whole database \hat{I}_{s_2} is the direct sum of (9) with respect to all blocks. Both relevant eigenvectors of \hat{G}_2 were found by in [4]:

$$\hat{G}_2 |\psi_2^\pm\rangle = \lambda_2^\pm |\psi_2^\pm\rangle, \quad \lambda_2^\pm = \exp[\pm 2i\theta_2], \quad |\psi_2^\pm\rangle = \frac{1}{\sqrt{2}} |t\rangle \pm \frac{i}{\sqrt{2}} |\text{ntt}\rangle \quad (10)$$

Here the $|\text{ntt}\rangle$ is a normalized sum of all non-target items in the target block:

$$|\text{ntt}\rangle = \frac{1}{\sqrt{b-1}} \sum_{\substack{x \neq t \\ \text{target block}}} |x\rangle, \quad \langle \text{ntt} | \text{ntt} \rangle = 1. \quad (11)$$

We shall need an angle θ_2 given by

$$\sin^2 \theta_2 = \frac{K}{N} = \frac{1}{b}. \quad (12)$$

The partial search algorithm of [6] creates a vector

$$|d\rangle = \hat{G}_1 \hat{G}_2^{j_1} \hat{G}_1^{j_0} |s_1\rangle. \quad (13)$$

see ¹. In the state $|d\rangle$ the amplitudes of all items in non-target blocks are zero. Notice that this algorithm uses global-local sequence of searches. We consider large blocks $b = N/K \rightarrow \infty$. The number of blocks K is an important parameter. We shall replace it with

¹We use a modification of [6], suggested in [8]

$$\sin \gamma = \frac{1}{\sqrt{K}}, \quad 0 \leq \gamma \leq \frac{\pi}{4} \quad (14)$$

The optimal version of this algorithm was find in [8]. It can be described by the following equations:

$$\cos(2j_1\theta_2) = \frac{\sin \gamma \cos 2\gamma}{\cos \gamma \sin 2\gamma}, \quad \tan(2j_0\theta_1) = \frac{\cos 2\gamma}{(\sin \gamma)\sqrt{3 - 4(\sin \gamma)^2}} \quad (15)$$

Partial search is faster then full search (1) by $\sim \sqrt{b}$, the coefficient in front of \sqrt{b} is explicitly calculated in [8] and studied as a function of number of blocks.

C. Notation and Setup for General Partial Search Algorithm

Let us introduce a unite vector:

$$|u\rangle = \frac{1}{\sqrt{b(K-1)}} \sum_{\substack{\text{all items in all} \\ \text{non-target blocks}}} |x\rangle, \quad \langle u|u\rangle = 1. \quad (16)$$

We shall use a three dimensional space. The orthonormal basis is formed by the target item $|t\rangle$, sum of all non-target items in the target block $|ntt\rangle$, defined in (11) and $|u\rangle$. All the state vectors involved in present quantum search problem can be written in this basis as

$$|V\rangle = (a, b, c)^T \quad (17)$$

meaning

$$|V\rangle = a|t\rangle + b|ntt\rangle + c|u\rangle \quad (18)$$

For example, the global uniform state which is used as initial state of searching is given by

$$|s_1\rangle = (\sin \gamma \sin \theta_2, \sin \gamma \cos \theta_2, \cos \gamma)^T \quad (19)$$

and the local uniform state is

$$|s_2\rangle = (\sin \theta_2, \cos \theta_2, 0)^T \quad (20)$$

The algorithms which we consider in this paper can be represented as matrices in this linear space. For example j_2 repetitions of the local iteration (8) is:

$$\hat{G}_2^{j_2} = \begin{pmatrix} \cos(2j_2\theta_2) & \sin(2j_2\theta_2) & 0 \\ -\sin(2j_2\theta_2) & \cos(2j_2\theta_2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (21)$$

The ordering of eigenvectors is $|t\rangle$, $|ntt\rangle$ and $|u\rangle$. The matrix has three eigenvectors:

$$\hat{G}_2^{j_2} |v_2^\pm\rangle = \exp(\pm 2i\theta_2 j_2) |v_2^\pm\rangle, \quad \hat{G}_2^{j_2} |v_2^0\rangle = |v_2^0\rangle \quad (22)$$

The eigenvectors can be represented as:

$$|v_2^\pm\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix}, \quad |v_2^0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (23)$$

Now let us turn our attention to global iterations (5), j_1 repetitions of the global iterations can be represented as

$$\hat{G}_1^{j_1} = \begin{pmatrix} \cos(2j_1\theta_1), & \sin(2j_1\theta_1) \sin \gamma, & \sin(2j_1\theta_1) \cos \gamma \\ -\sin(2j_1\theta_1) \sin \gamma, & (-1)^{j_1} \cos^2 \gamma + \cos(2j_1\theta_1) \sin^2 \gamma, & \sin \gamma \cos \gamma ((-1)^{j_1+1} + \cos(2j_1\theta_1)) \\ -\sin(2j_1\theta_1) \cos \gamma, & \sin \gamma \cos \gamma ((-1)^{j_1+1} + \cos(2j_1\theta_1)), & (-1)^{j_1} \sin^2 \gamma + \cos(2j_1\theta_1) \cos^2 \gamma \end{pmatrix} \quad (24)$$

This is a simplified asymptotic expression valid in the limit of large blocks $b \rightarrow \infty$. We used (14). The matrix has three eigenvectors:

$$\hat{G}_1^{j_1} |v_1^\pm\rangle = \exp(\pm 2i\theta_1 j_1) |v_1^\pm\rangle, \quad \hat{G}_1^{j_1} |v_1^0\rangle = (-1)^{j_1} |v_1^0\rangle \quad (25)$$

The eigenvectors can be represented as:

$$|v_1^\pm\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \sin \gamma \\ \pm i \cos \gamma \end{pmatrix}, \quad |v_1^0\rangle = \begin{pmatrix} 0 \\ \cos \gamma \\ -\sin \gamma \end{pmatrix}. \quad (26)$$

D. General Problem of Partial Search

The ultimate goal of partial search is to start with the uniform state $|s_1\rangle$ and locate the target block, and obviously GRK is not the only means to achieve the goal. Based on the two types of queries, global and local iterations, we naturally generalize GRK into a wide set of partial search algorithms by alternate use of the two iterations:

$$\hat{G}(j_k, j_{k-1}, \dots, j_2, j_1, j_0) = \hat{G}_1^{j_k} \hat{G}_2^{j_{k-1}} \hat{G}_1^{j_{k-2}} \hat{G}_2^{j_{k-3}} \dots \hat{G}_1^{j_2} \hat{G}_2^{j_1} \hat{G}_1^{j_0} \quad (27)$$

which fulfills the following condition

$$\langle u | \hat{G}(j_k, j_{k-1}, \dots, j_2, j_1, j_0) | s_1 \rangle = 0 \quad (28)$$

This means that amplitude of each item in non-target block is zero. Here all j_i are non-negative integers. The total number of queries for these algorithms is given by $S = \sum_{i=0}^k j_i$, and we should try to minimize S to find the optimal one.

Now let's consider various sequences. A few discussions can be made here:

- 1) The last queries in the sequence should be the global iterations. Note that local iteration (21) doesn't do anything on $|u\rangle$ but only rotates state component inside the target block, so if an algorithm makes computational state in the target block after the last local queries, it can simply waive the last local queries since the state must already rest in the target block before those unnecessary last local iterations. That is, if $\langle u | \hat{G}_2^{j_{k+1}} \hat{G}_1^{j_k} \hat{G}_2^{j_{k-1}} \dots \hat{G}_1^{j_2} \hat{G}_2^{j_1} \hat{G}_1^{j_0} | s_1 \rangle = 0$ then we must also have $\langle u | \hat{G}_1^{j_k} \hat{G}_2^{j_{k-1}} \dots \hat{G}_1^{j_2} \hat{G}_2^{j_1} \hat{G}_1^{j_0} | s_1 \rangle = 0$.
- 2) The first queries could be either local or global iterations. Sequences starting with the global ($j_0 > 0$) include: $\hat{G}_1^{j_0}$ (Global), $\hat{G}_1^{j_2} \hat{G}_2^{j_1} \hat{G}_1^{j_0}$ (Global-Local-Global), and so on. The simplest one, with only global queries involved, gives nothing but the original Grover's full search (1), which saves no steps but precisely locates the target item. The next simplest one, Global-Local-Global, will be studied in later section. Note that the up-to-now established optimal partial search GRK algorithm also falls into this category, with $j_2 = 1$ and j_0, j_1 specified by (15).
- 3) For those starting with local queries ($j_0 = 0$ and $j_1 > 0$), we have: $\hat{G}_1^{j_2} \hat{G}_2^{j_1}$ (Local-Global), $\hat{G}_1^{j_4} \hat{G}_2^{j_3} \hat{G}_1^{j_2} \hat{G}_2^{j_1}$ (Local-Global-Local-Global), and so on. We will later fully consider the Local-Global algorithm and also discuss one particular case of Local-Global-Local-Global which has j_4 to be one, namely with only one global query applied after the last local iterations (we call this as Local-Global-Local sequence).
- 4) The argument based on the optimality of Grover's full search puts lower limit of the partial search steps $S \geq \frac{\pi}{4}\sqrt{N} - \frac{\pi}{4}\sqrt{b}$. It was shown in [6] that $S = \frac{\pi}{4}\sqrt{N} - R\sqrt{b}$. Here R is a constant independent of b . The optimal scheme we are seeking should maximize R . Also according to this thought, we expect the total number of global queries should scale asymptotically as $\frac{\pi}{4}\sqrt{N} - \eta\sqrt{b}$, since in partial search we should be faster than full search $\frac{\pi}{4}$ but save steps only of order \sqrt{b} . As for total number of local queries it should scale like $\alpha\sqrt{b}$, since $\pi\sqrt{b}$ times local iteration will have rotated the state vector a whole lap on the target block $|t\rangle \rightarrow -|ntt\rangle$ plane. It follows from these consideration that $R = \eta - \alpha$. Though the numbers of local and global iterations j_i are integer numbers and in principle not continuous variables, in the large block limit we can reasonably treat them as quasi-continuous and use them as function arguments.
- 5) R is a function of block number K or instead the parameter $\gamma = \arcsin(\frac{1}{\sqrt{K}})$ with $0 < \gamma \leq \pi/4$. Another limit we will consider is the large K limit, $K \gg 1$ or equivalently $\gamma \rightarrow 0$. In this limit the mathematical formulae will be significantly simplified, which will be very helpful to analytical efforts as will be seen in later sections. The cases with not so large K could be complemented by direct numerical verification.

III. LOCAL-GLOBAL SEQUENCE OF SEARCHES

The Local-Global sequence of searches is the simplest partial search scheme. For such a sequence $\hat{G}(j_2, j_1)$ with first j_1 local iterations applied and then j_2 global, we have the final computational state to be:

$$\hat{G}_1^{j_2} \hat{G}_2^{j_1} |s_1\rangle = (A, B, C)^T \quad (29)$$

Here A,B,C are the coefficients of components $|t\rangle, |ntt\rangle, |u\rangle$ respectively (see (18)), which are given by

$$\begin{aligned} A &= \sin \gamma \cos(2j_2\theta_1) \sin(2j_1\theta_2) + \sin^2 \gamma \sin(2j_2\theta_1) \cos(2j_1\theta_2) + \cos^2 \gamma \sin(2j_2\theta_1) \\ B &= -\sin^2 \gamma \sin(2j_2\theta_1) \sin(2j_1\theta_2) + (-)^{j_2} \cos^2 \gamma \sin \gamma \cos(2j_1\theta_2) \\ &\quad + \sin^3 \gamma \cos(2j_2\theta_1) \cos(2j_1\theta_2) + \sin \gamma \cos^2 \gamma (\cos(2j_2\theta_1) - (-)^{j_2}) \\ C &= -\sin \gamma \cos \gamma \sin(2j_2\theta_1) \sin(2j_1\theta_2) + \sin^2 \gamma \cos \gamma (\cos(2j_2\theta_1) - (-)^{j_2}) \cos(2j_1\theta_2) \\ &\quad + (-)^{j_2} \sin^2 \gamma \cos \gamma + \cos^3 \gamma \cos(2j_2\theta_1) \end{aligned} \quad (30)$$

Then to accomplish the partial search we should have the constraint equation

$$\langle u | \hat{G}_1^{j_2} \hat{G}_2^{j_1} | s_1 \rangle = C = 0 \quad (31)$$

which means that the amplitude of every item in non-target block vanish (28). Since there are only one set of local and one set of global iterations, we apply the scaling as discussed before and introduce

$$j_1 = \alpha\sqrt{b}, \quad j_2 = \frac{\pi}{4}\sqrt{N} - \eta\sqrt{b} \quad (32)$$

We have the limitation $0 \leq \alpha < \pi$ and $\eta < \frac{\pi}{4}\sqrt{K}$. Now we can rewritten the constraint equation as following:

$$\cos \gamma \cdot \left([\sin^2 \gamma \cos(2\alpha) + \cos^2 \gamma] \sin\left(\frac{2\eta}{\sqrt{K}}\right) - \sin \gamma \sin(2\alpha) \cos\left(\frac{2\eta}{\sqrt{K}}\right) + (-)^{j_2} \sin^2 \gamma [1 - \cos(2\alpha)] \right) = 0 \quad (33)$$

Note that $K \geq 2$ so $\cos \gamma > 0$, hence it drops out in the above equation and we have

$$F(\alpha, \eta) = [\sin^2 \gamma \cos(2\alpha) + \cos^2 \gamma] \sin\left(\frac{2\eta}{\sqrt{K}}\right) - \sin \gamma \sin(2\alpha) \cos\left(\frac{2\eta}{\sqrt{K}}\right) + (-)^{j_2} \sin^2 \gamma [1 - \cos(2\alpha)] = 0 \quad (34)$$

Remember we are interested in the fastest algorithm (using the fewest queries), especially those using less steps than Grover's full search. Now we have the total number of queries to be $S = \frac{\pi}{4}\sqrt{N} - (\eta - \alpha)\sqrt{b}$, thus to minimize S we should maximize $R = \eta - \alpha$ under the constraint (34). The point here is that from the constraint we can consider η as a function of α , thus R is also a function of α , to which it should be optimized. We then have $\frac{dR}{d\alpha} = -\frac{d\eta}{d\alpha} + 1 = 0$, which leads to

$$\sin(2\gamma) [\cos^2 \gamma (1 - \cos(2\alpha) \cos(\frac{2\eta}{\sqrt{K}})) + (-)^{j_2} \sin \gamma \sin(2\alpha)] = 0 \quad (35)$$

Again since $0 < 2\gamma \leq \pi/2$, we have

$$\sin(2\gamma) > 0 \quad (36)$$

so we simplify the above equation as

$$H(\alpha, \eta) = 2 \sin \alpha [\cos^2 \gamma \sin \alpha \cos(\frac{2\eta}{\sqrt{K}}) + (-)^{j_2} \sin \gamma \cos \alpha] = 0 \quad (37)$$

There is an apparent solution for (34) and (37), namely $\sin \alpha = \sin(\frac{2\eta}{\sqrt{K}}) = 0$. This gives $\alpha = 0$ and $\eta = l \cdot \pi \cdot \sqrt{K}$, $l = 0, -1, -2, \dots$, but to maximize $R = \eta - \alpha$ we should have $\alpha = \eta = 0$, which again recovers the Grover's full search solution and is a trivial one for partial search. In the following we only consider nontrivial solutions with $\sin \alpha \neq 0$.

Let's first look at the large K or small γ limit. In leading order, the two equations (34) and (37) are reduced to be

$$F(\alpha, \eta) = \sin(\frac{2\eta}{\sqrt{K}}) - \gamma \sin(2\alpha) \cos(\frac{2\eta}{\sqrt{K}}) + \gamma^2 (-1)^{j_2} 2 \sin^2 \alpha = 0 \quad (38)$$

$$H(\alpha, \eta) = \sin \alpha [\sin \alpha \cos(\frac{2\eta}{\sqrt{K}}) + (-1)^{j_2} \gamma \cos \alpha] = 0 \quad (39)$$

To satisfy (38), the first term $\sin(\frac{2\eta}{\sqrt{K}})$ must be at least as small as $\sim \gamma$, so in leading order we must have $\cos(\frac{2\eta}{\sqrt{K}}) \sim 1$. This, combined with (39), requires also $\sin \alpha$ must be at least as small as $\sim \gamma$ (we don't consider $\sin \alpha = 0$ as mentioned before), which means $\cos \alpha \sim 1$. So finally we reduce (39) in leading order to be

$$\sin \alpha + (-1)^{j_2} \gamma = 0 \quad (40)$$

Remember we have $0 \leq \alpha < \pi$ thus $\sin \alpha \geq 0$, so the above equation has solution ONLY for ODD values of j_2 . For odd j_2 the solution is $\sin \alpha = \gamma$, this yields two solutions $\alpha = \gamma$ and $\alpha = \pi - \gamma$. By substituting $\sin \alpha = \gamma$ back into (38) we get in leading order $\sin \frac{2\eta}{\sqrt{K}} = 2\gamma^2$, which means $\frac{2\eta}{\sqrt{K}} = 2\gamma^2 + 2l\pi$ or $\frac{2\eta}{\sqrt{K}} = -2\gamma^2 + (2l-1)\pi$, $l = 0, -1, -2, \dots$. But again remember our goal is to maximize $R = \eta - \alpha$, so we adopt the optimal Local-Global solution $\alpha = \gamma$ and $\eta = \gamma$ in large K limit. This, however, seems giving no speedup compared with full search since $R = \eta - \alpha \sim 0$. To clarify this, we need go to higher order of equations (38) and (39). By properly including corrections up to $\sim \gamma^3$ we can find the solution to be

$$\alpha = \gamma + \frac{1}{2}\gamma^3 + o(\gamma^4) \quad , \quad \eta = \gamma + \frac{2}{3}\gamma^3 + o(\gamma^4) \quad (41)$$

Now we see in large K limit, the Local-Global sequences of search do achieve speedup

$$R = \frac{1}{6}\gamma^3 + o(\gamma^4) \quad (42)$$

which is very small but still nonzero.

Let's now turn back to finite values of K . From (37) we have

$$\cos(\frac{2\eta}{\sqrt{K}}) = \frac{(-1)^{j_2+1} \sin \gamma \cos \alpha}{\cos^2 \gamma \sin \alpha} \quad (43)$$

Combine (43) and (34) together we get

$$\cos(\frac{2\eta}{\sqrt{K}}) = \frac{\sin \gamma \cos \alpha}{\cos^2 \gamma \sin \alpha} \quad , \quad \sin(\frac{2\eta}{\sqrt{K}}) = \frac{2 \sin^2 \gamma (\cos^2 \gamma \sin^2 \alpha + \cos^2 \alpha)}{\cos^2 \gamma [\sin^2 \gamma \cos(2\alpha) + \cos^2 \gamma]} \quad (44)$$

By requiring $\cos^2(\frac{2\eta}{\sqrt{K}}) + \sin^2(\frac{2\eta}{\sqrt{K}}) = 1$ we obtain the equation determining the value of α and from α we can obtain η . Generally it is hard to analyze the problem analytically, so we proceed to numerical solutions, see the figures Fig.1,2. In these figures we plot R as a function of α with different K for both even and odd values of j_2 , at each point η is determined from the constraint (34). For the odd j_2 case there is always a positive maximum which is faster than full search, and as K approaches very large values, the maximum of the ϵ v.s. α curve moves gradually toward the origin which stands for the full search solution. But for even j_2 , R is always negative, hence slower than full search.

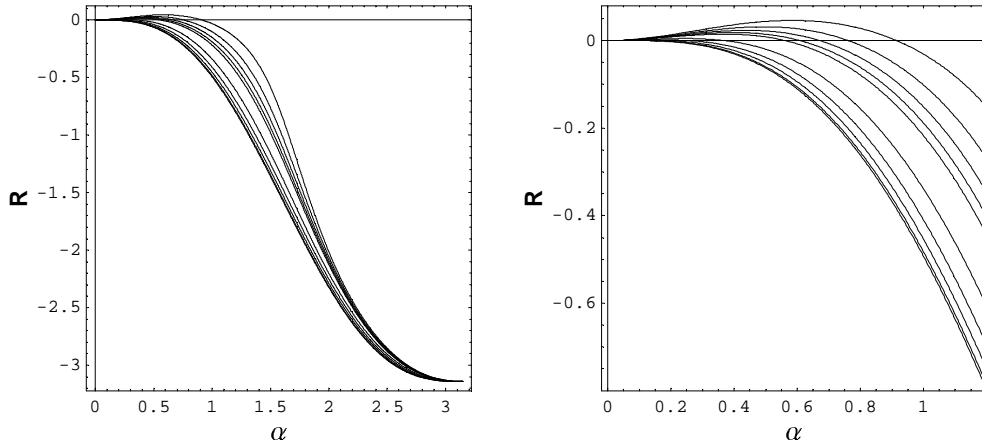


FIG. 1. Dependence of R on α with odd j_2 in Local-Global sequence, the curves from top to bottom are for $K = 5, 6, 7, 8, 9, 18, 36, 72, 144, 200$ respectively. For each α the value of η is solved from (34). Right panel is the amplification of the area near origin in Left panel.

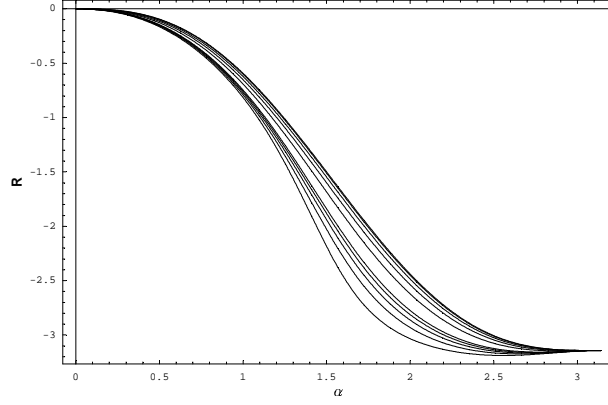


FIG. 2. Dependence of R on α with even j_2 in Local-Global sequence, the curves from top to bottom are for $K = 5, 6, 7, 8, 9, 18, 36, 72, 144, 200$ respectively. For each α the value of η is solved from (34).

An important comparison is to be made between this Local-Global algorithm and the established GRK one. Our numeric results show that the odd j_2 Local-Global searches can get $R \approx 0.3$ with $K = 2, 3$ and get $0 < R < 0.09$ for $K \geq 4$. For GRK, however, it can achieve $R > 0.32$ for all $K \geq 2$. So the GRK is much faster than the present Local-Global searches.

To sum up with the Local-Global sequence of searches, we find it can be faster than full search with $R\sqrt{b}$ speedup for block number K up to several tens, but it is always slower than the GRK optimized partial search.

IV. GLOBAL-LOCAL-GLOBAL SEQUENCE OF SEARCHES

In this section we shall consider another algorithm for partial search, the Global-Local-Global sequence of searches, which starts with (4), first applies j_0 global iterations (5), then j_1 local iterations (8) and finally j_2 global iterations:

$$\hat{G}_1^{j_2} \hat{G}_2^{j_1} \hat{G}_1^{j_0} |s_1\rangle. \quad (45)$$

Again amplitudes of all items in non-target blocks should vanish at the end of algorithm:

$$\langle u | \hat{G}_1^{j_2} \hat{G}_2^{j_1} \hat{G}_1^{j_0} |s_1\rangle = 0. \quad (46)$$

Let us present this equation in an explicit form. After first set of global iterations the state of the database will be:

$$\hat{G}_1^{j_0} |s_1\rangle = \frac{\sin \gamma}{\sqrt{b}} \begin{pmatrix} \cos(2j_0\theta_1) \\ -\sin(2j_0\theta_1) \sin \gamma \\ -\sin(2j_0\theta_1) \cos \gamma \end{pmatrix} + \begin{pmatrix} \sin(2j_0\theta_1) \\ \sin \gamma \cos(2j_0\theta_1) \\ \cos \gamma \cos(2j_0\theta_1) \end{pmatrix}. \quad (47)$$

After local iterations the state of the database is:

$$\hat{G}_2^{j_1} \hat{G}_1^{j_0} |s_1\rangle = \begin{pmatrix} \sin(2j_0\theta_1) \cos(2j_1\theta_2) + \sin \gamma \cos(2j_0\theta_1) \sin(2j_1\theta_2) \\ -\sin(2j_0\theta_1) \sin(2j_1\theta_2) + \sin \gamma \cos(2j_0\theta_1) \cos(2j_1\theta_2) \\ \cos \gamma \cos(2j_0\theta_1) \end{pmatrix}. \quad (48)$$

Here we neglected $\sim 1/\sqrt{b}$ terms (namely taking large block limit). After next set of global searches we have to calculate only third component (coefficient of $|u\rangle$) of the vector:

$$\begin{aligned} 0 &= \langle u | \hat{G}_1^{j_2} \hat{G}_2^{j_1} \hat{G}_1^{j_0} |s_1\rangle \\ &= \cos(2j_1\theta_2) \{ -\cos \gamma \sin(2j_2\theta_1) \sin(2j_0\theta_1) + \sin^2 \gamma \cos(2j_0\theta_1) \cos \gamma [(-1)^{j_2+1} + \cos(2j_2\theta_1)] \} + \\ &\quad \sin(2j_1\theta_2) \{ -\sin \gamma \cos \gamma \sin(2j_2\theta_1) \cos(2j_0\theta_1) - \sin \gamma \cos \gamma \sin(2j_0\theta_1) [(-1)^{j_2+1} + \cos(2j_2\theta_1)] \} + \\ &\quad + \cos \gamma \cos(2j_0\theta_1) [(-1)^{j_2} \sin^2 \gamma + \cos^2 \gamma \cos(2j_2\theta_1)] \end{aligned} \quad (49)$$

This constraint equation guarantees the amplitudes of all items in all non-target blocks vanish to successfully complete the partial search.

First let us check the case $j_1 = 0$, no local searches. In this case the constraint equation can be reduced to $\cos[2\theta_1(j_0 + j_2)] = 0$. This is just the full search: we use $j_0 + j_2 = \frac{\pi}{4}\sqrt{N}$ global iterations to find the target item.

Now let us consider more general and complicated case. We expect the following scaling, namely the global iterations $j_0 = \frac{\pi}{4}\sqrt{N} - \eta\sqrt{b}$, $j_2 = \beta\sqrt{b}$ and total local iterations $j_1 = \alpha\sqrt{b}$ with $\eta \leq \frac{\pi}{4}\sqrt{K}$, $0 \leq \beta < \frac{\pi}{4}\sqrt{K}$ and $0 \leq \alpha < \pi$. It should be remembered that our purpose is to minimize $j_0 + j_1 + j_2 = \frac{\pi}{4}\sqrt{N} - R\sqrt{b}$, $R = \eta - \beta - \alpha$, or equivalently maximize R . Our strategy is similar to that used for analyzing Local-Global sequence: study the large K limit analytically while deal with finite K case numerically.

In the large K or small γ limit, we can take the leading order of (49) and simplify our constraint to a much simpler form :

$$\eta = \beta \cos(2\alpha) + \frac{1 - (-)^{j_3}}{2} \sin(2\alpha) \quad (50)$$

From this equation, there are two possibilities:

- 1) j_2 is even, thus $\eta = \beta \cos(2\alpha)$. In this case the problem simplifies into maximizing $R = \beta(\cos(2\alpha) - 1) - \alpha$ with $0 \leq \beta < \frac{\pi}{4}\sqrt{K}$ and $0 \leq \alpha < \pi$. Note that $\cos(2\alpha) - 1 \leq 0$ so the maximum of R must occur at $\eta = \beta = \alpha = 0$, which is again the trivial full search solution (1).
- 2) j_2 is odd, thus $\eta = \beta \cos(2\alpha) + \sin(2\alpha)$. We then have to maximize $R = \beta(\cos(2\alpha) - 1) + \sin(2\alpha) - \alpha$ with $0 \leq \beta < \frac{\pi}{4}\sqrt{K}$ and $0 \leq \alpha < \pi$. The solution is

$$\eta = \frac{\sqrt{3}}{2}, \quad \beta = 0, \quad \alpha = \pi/6, \quad (51)$$

which has achieved $R\sqrt{b} \approx 0.3424\sqrt{b}$ speedup with respect to full search. This nontrivial optimal solution is exactly the GRK algorithm, with vanishingly small odd j_2 namely $j_2 = 1$.

Now let us discuss finite values of K . We calculated dependence of R on β by determining α from constraint equations and optimizing η numerically at each value of β . We done this numerically for up to 200 blocks. The dependence of R on β is monotonous. Corresponding figures took too much memory, so we withdraw them from the paper. For even j_2 searches we consider are slower than full Grover search, while odd j_2 these searches are faster than full search. The optimal searches always occur with odd and vanishing j_2 (the GRK case), being about $0.34\sqrt{b}$ faster than full search. The large K case also confirm our analysis above. Our results here also confirm that the optimum partial search is GRK [8], it has the form $\hat{G}_1 \hat{G}_2^{j_1} \hat{G}_1^{j_0}$.

To conclude, we have found that Global-Local-Global sequence of searches can be much faster than full search (1) with odd times global iterations applied in the end, and the optimal search is determined to be the GRK algorithm for all values of K .

V. LOCAL-GLOBAL-LOCAL SEQUENCE OF SEARCHES

In this section we shall discuss another algorithm of partial search. It belongs to the category of Local-Global-Local-Global sequence but with only one global query applied in the end, which we call as Local-Global-Local sequence. We start with (4) first apply j_1 local iterations, then j_2 global iterations (8) and then j_3 local iterations, and eventually a single global query:

$$\hat{G}_1 \hat{G}_2^{j_3} \hat{G}_1^{j_2} \hat{G}_2^{j_1} |s_1\rangle. \quad (52)$$

One particular advantage of this type of sequence is that when j_1 tends to zero our Local-Global-Local will degenerate to the GRK type algorithm. Amplitudes of all items in non-target blocks should vanish at the end:

$$\langle u | \hat{G}_1 \hat{G}_2^{j_3} \hat{G}_1^{j_2} \hat{G}_2^{j_1} | s_1 \rangle = 0. \quad (53)$$

As before, we introduce the scaling of iteration numbers to be $j_1 = \alpha\sqrt{b}$, $j_3 = \delta\sqrt{b}$, and $j_2 = \frac{\pi}{4}\sqrt{N} - \eta\sqrt{b}$. The total number of queries will be $S = \frac{\pi}{4}\sqrt{N} - R\sqrt{b}$ with $R = \eta - \alpha - \delta$ which we want minimize under constraint (53). The explicit form of the above constraint equation is as follows:

$$0 = X \cdot \sin(2\alpha) + Y \cdot [\cos(2\alpha) - 1] + Z \quad (54)$$

$$X = \sin \gamma \sin(2\gamma) \sin(2\delta) \sin\left(\frac{2\eta}{\sqrt{K}}\right) + \sin^2 \gamma \sin(2\gamma) \cos(2\delta) \cos\left(\frac{2\eta}{\sqrt{K}}\right) + \sin \gamma \cos \gamma \cos(2\gamma) \cos\left(\frac{2\eta}{\sqrt{K}}\right)$$

$$Y = \sin^2 \gamma \sin(2\gamma) \sin(2\delta) \cos\left(\frac{2\eta}{\sqrt{K}}\right) - \sin^3 \gamma \sin(2\gamma) \cos(2\delta) \sin\left(\frac{2\eta}{\sqrt{K}}\right) - \sin^2 \gamma \cos \gamma \left[\cos(2\gamma) \sin\left(\frac{2\eta}{\sqrt{K}}\right) + (-)^{j_2} \right]$$

$$Z = \sin(2\gamma) \sin(2\delta) \cos\left(\frac{2\eta}{\sqrt{K}}\right) - \sin \gamma \sin(2\gamma) \cos(2\delta) \sin\left(\frac{2\eta}{\sqrt{K}}\right) - \cos \gamma \cos(2\gamma) \sin\left(\frac{2\eta}{\sqrt{K}}\right)$$

Though the above equation is complicated, it is very easy to solve numerically. Also taking large K limit can significantly simplify it. So as we did before, we analytically study the large K limit, complemented by numerical results from very small K to very large K .

By taking large K or small γ limit, we obtain from leading order of (54) the following:

$$\sin(2\alpha) + 2 \sin(2\delta) - 2\eta = 0 \quad (55)$$

We then have $\eta = \frac{1}{2} \sin(2\alpha) + \sin(2\delta)$ and hence the total number of queries to be

$$R = \frac{1}{2} \sin(2\alpha) - \alpha + \sin(2\delta) - \delta \quad (56)$$

By requiring $\frac{\partial R}{\partial \alpha} = \frac{\partial R}{\partial \delta} = 0$ we get the solution maximizing R

$$\alpha = 0, \quad \delta = \frac{\pi}{6}, \quad \eta = \frac{\sqrt{3}}{2} \quad (57)$$

This, with zero j_1 , again recovers the GRK optimized solution. So in large K limit we see Local-Global-Local sequence is no faster than GRK algorithm. Different from Local-Global and Global-Local-Global, here in Local-Global-Local we notice that the odd j_2 and even j_2 converge to each other when approaching the optimal solution and the oscillation terms in (54) with factor $(-)^{j_2}$ disappear.

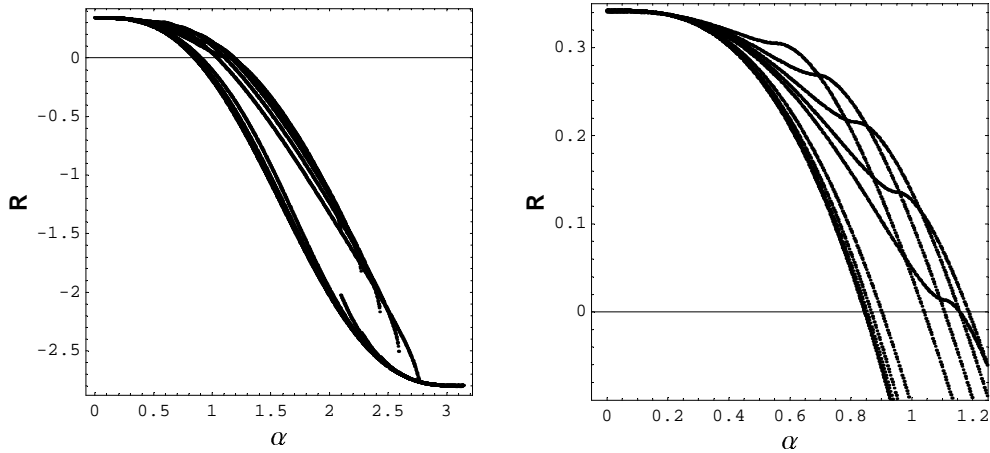


FIG. 3. Dependence of R on α with odd j_2 in Local-Global-Local sequence, the curves from top to bottom are for $K = 5, 6, 7, 8, 9, 18, 36, 72, 144, 200$ respectively. For each α the value of η is solved from (54) with δ optimized numerically. Right panel is the amplification of the area near origin in Left panel.

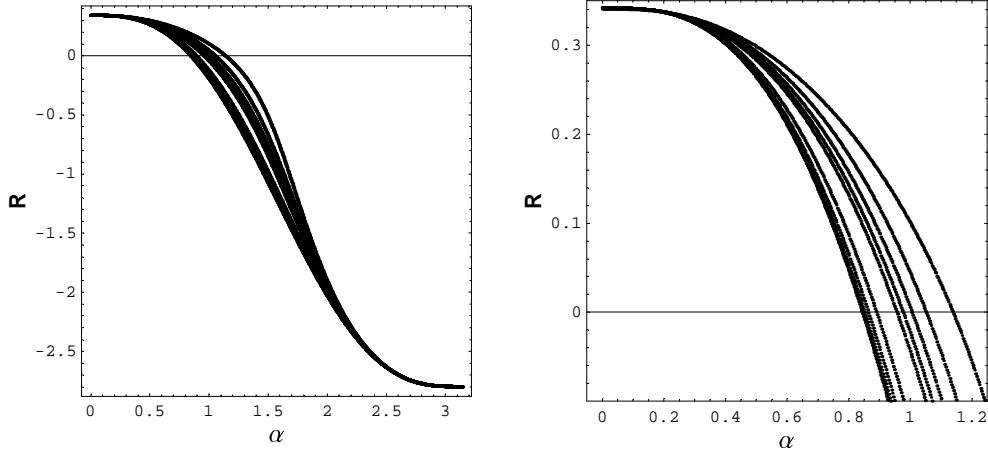


FIG. 4. Dependence of R on α with even j_2 in Local-Global-Local sequence, the curves from top to bottom are for $K = 5, 6, 7, 8, 9, 18, 36, 72, 144, 200$ respectively. For each α the value of η is solved from (54) with δ optimized numerically.

For general values of K , we conduct the numerical method and show the results in figures Fig.34, which plot R as a function of α with η solved from (54) and δ optimized numerically. As can be seen, the optimal solutions always occur with $\alpha = 0$ which goes back to GRK case. Also we note that even and odd j_2 give same results around optimal point.

So in this section we have established that the Local-Global-Local sequence of searches can be much faster than full search, but is no faster than GRK algorithm. In the appendix an alternative approach for Local-Global-Local sequence based on a conjecture about cancellation of oscillation terms in (54) is briefly described, which though is not directly relevant here but arrives at similar result and may shed light for future exploration of even more complicated sequences.

VI. SUMMARY

We considered different partial search algorithms, which consists of a sequence of local and global searches. We introduced a general framework for studying partial quantum search algorithms and classified various possible sequences. Particularly, we studied the Local-Global, Global-Local-Global as well as Local-Global-Local sequences of searches by combining numerical study for wide range values of K and analytical results for large K limit. All these algorithms achieve \sqrt{b} speedup compared to the Grover's full quantum search. GRK algorithm [8] is the fastest among partial search algorithms, which we considered.

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APPENDIX: APPENDIX: REMARKS ON LOCAL-GLOBAL-LOCAL SEQUENCE

Let us make two remarks:

1) Let's start from the explicit form of constraint equation (54). We minimizing $j_1 + j_2 + j_3$. It is interesting to note that at the minimum the oscillation terms cancel. The coefficient at $(-1)^{j_2}$ vanishes because of GRK equation:

$$\cos \gamma \sin 2\gamma \cos(2\theta_2 j_3) = \sin \gamma \cos 2\gamma \quad (\text{A1})$$

This is exactly the first equation of (15). We also can write it in the form:

$$\cos(2j_3\theta_2) = \frac{1 - \tan^2 \gamma}{2}, \quad \sin(2j_3\theta_2) = \frac{\sqrt{3 - 4 \sin^2 \gamma}}{2 \cos^2 \gamma} \quad (\text{A2})$$

2) Minimum number of iterations for Local-Global-Local sequences corresponds to $j_1 = 0$ case and the algorithm is reduced back to GRK version of partial search, see small j_1 increase $j_2 + j_1$ very little since $d(j_2 + j_1)/dj_1 = 0$ and $d^2(j_2 + j_1)/dj_1^2 = 0$ at $j_1 = 0$.

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